On Generalized Hilbert Algebra

by

SUNG Li-Yeng

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The undersigned certify that we have read a thesis, entitled "On Generalized Hilbert Algebra", which was submitted to the Graduate School by Mr. SUNG Li-Yeng (宋立言) in partial fulfillment of the requirement for the degree of Master of Philosophy in Mathematics. We recommend that it be accepted.

Prof. Donald Bures

Dr. P.K. Tam

Dr. Y.C. Wong

Dr. K.F. Ng

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1. Introduction

The theory of Hilbert algebra is a powerful tool in dealing with semi-finite von Neumann algebras. The notion of Hilbert algebra was generalized by J. Dixmier to the quasi-unitary algebras in 1952. Later, Tomita developed the concepts of modular Hilbert algebras and generalized Hilbert algebras. Tomita's original papers are very difficult and it was Takesaki who clarified his ideas in the now well-known lecture notes [11] in 1970.

However, the expositions in [11] are maninly on modular Hilbert algebras and theorems concerned with generalized Hilbert algebras are proved only by means of more involved modular Hilbert algebras, for example corollary 9.1, th. 10.1, 12.3, 13.1, 13.2 14.1 and 14.2 in [11] .

In 1973, A. Van Daele gave a new approach to the Tomita-Takesaki Theory ([12]) in which the above-mentioned corollary 9.1. and th. 10.1 were directly proved by an ingenious method.

The important theorem of Tomita on the commutation of tensor products of von Neumann algebras ([11], th. 12.3) was also proved by M. A. Rieffel and A. Van Dael with a "geometric" argument in 1974. (see [9]). It turns out that the "geometric" idea employed there by them leads to yet another bounded operator approach to the Tomita-Takesaki Theory (see [10]) .

The development of the theory of weights makes it possible to prove the remaining th. 13.1, 13.2, 14.1 and 14.2 of [11] in the setting of generalized Hilbert algebra and more general results can be obtained (see [1], [2], [3], [7] and [8]).

Thus, in retrospect, we see that almost all of the results in [16] can be obtained within the framework of generalized Hilbert algebra.

In section 2 of this thesis, we give another proof of the Tomita Theorem on tensor product which is "non-geometric" and in the same time making no use of the modular Hilbert algebra. A characterization of the von Neumann algebra of a left Hilbert algebra by the existence of a certain involutive isometry is given in section 3. The proof of the uniqueness of the automorphism group satisfying the K.M.S. condition with respect to a normal faithful semi-finite weight in [2] is intrinsically involved with the modular Hilbert algebra. We prove the same uniqueness theorem in section 4 by a modification of the ideas in [10] which is more direct in the sense that it does not involve with the notion of modular Hilbert algebra.

Before going on to section 2, we would like to write down the definition of a left Hilbert algebra and the relevant basic results that are going to be used frequently in the sequel.

<u>Definition 1.1</u> Let U be an involutive algebra over the complex number field $\mathbb{C}$ with involution $\xi \in U \longmapsto \xi^{\#} \in U$. U is called a left Hilbert ālgebra if U admits an inner product $(\xi|\eta)$ and satisfies the following conditions :

(I) $(\xi\eta|\zeta) = (\eta|\xi^{\#}\zeta)$;

(II) For each $\xi \in U$, the map :
$\eta \in U \longmapsto \xi\eta \in U$ is continuous ;

(III) The subalgebra U^2 of U, spanned by the element $\xi\eta$ with $\xi, \eta \in U$, is dense in U ;

(IV) The involution : $\xi \in U \longrightarrow \xi^{\#} \in U$ is preclosed as a real linear operator on the real pre-Hilbert space U.

The closure of the involution in H, the completion of U, is denoted by S. S is a densely defined closed operator with a densely defined adjoint operator F. FS is therefore a positive self-adjoint operator on H, which we shall denote by Δ. S has a polar decomposition $S = J\Delta^{\frac{1}{2}}$, where J is an involutive isometry. Some elementary properties of S, J and Δ are summarized in the following lemma.

<u>Lemma 1.2</u> There exist an isometric involution J and a positve self-adjoint operator Δ such that

(i) $\mathscr{D}^{\#} = \mathscr{D}(\Delta^{\frac{1}{2}})$ and $\mathscr{D}^{b} = \mathscr{D}(\Delta^{-\frac{1}{2}})$

(ii) $J\Delta J = \Delta^{-1}$ and $Jf(\Delta)J = \bar{f}(\Delta^{-1})$ for every measurable function f defined on the open interval $(0,\infty)$; in particular, $J\Delta^{it} = \Delta^{it}J$;

(iii) $S = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J$,

$F = J\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}J$;

(iv) $\Delta = FS$ and $\Delta^{-1} = SF$.

Proof : ([11] , th. 7.1)

Q. E. D.

The operator Δ is called the modular operator of U and J the unitary involution of U .

For each $\xi \in U$ the bounded operator on H which is the extension of $\eta \in U \longmapsto \xi\eta \in U$ is denoted by $\Pi(\xi)$.

It is clear that Π is a non-degenerate * - representation of U . The von Neumann algebra generated by $\Pi(U)$ on H is denoted by $L(U)$, called the left von Neumann algebra of U .

$\eta \in D(F)$ is called Π' - bounded if the map $\xi \longrightarrow \Pi(\xi)\eta$ is continuous on U and the bounded operator extensions of this map is denoted $\Pi'(\eta)$.

The set of all Π' - bounded elements of H is denoted by U', which is also a left Hilbert algebra and we can therefore define U'' . It is clear that $U \subseteq U''$ and if $U = U''$, U is said to be achieved.

<u>Lemma 1.3</u> $L(U)'$ is generated by $\Pi'(U')$

<u>Proof</u>. ([11] , th. 3.1)

Q. E. D.

<u>Lemma 1.4</u> Let U be a left Hilbert algebra.

Then $J\,L(U)J = L(U)'$ and $\Delta^{it}\,L(U)\Delta^{-it} = L(U)$ for all $t \in \mathbb{R}$.

<u>Proof</u>. ([12] , th. 5.1)

Q. E. D.

<u>Lemma 1.5</u> Let U be an achieved left Hilbert algebra. If $\xi \in U$, then $J\xi \in U'$ and $\Pi'(J\xi) = J\Pi(\xi)J$. If $\xi \in U$, then $\Delta^{it}\xi \in U$ for all $t \in \mathbb{R}$ and $\Pi(\Delta^{it}\xi) = \Delta^{it}\,\Pi(\xi)\Delta^{-it}$. A similar statement holds for U' .

<u>Proof</u>. ([12] , th. 5.3)

Q. E. D.

By lemma 1.5 , $\xi \longmapsto \Delta^{it}\xi$ is therefore an algebraic isomorphism on an achieved left Hilbert algebra which induces a automorphism group $\Delta^{it}(\cdot)\Delta^{-it}$ on $L(U)$. $\Delta^{it}(\cdot)\Delta^{-it}$ is called the modular automorphism group of $L(U)$.

<u>Example 1.6</u> Let M be a von Neumann algebra on H and ω be a cyclic and separating vector of M . A left Hilbert algebra on H can be defined in the following way.

$$U = \{ A\omega : A \in M \}$$

$$(A\omega)^{\#} = A^{*}\omega$$

The scalar product on U is just that of H .

It can be shown that U is achieved, $U' = \{A'\omega : A \in M\}$, $F(A'\omega) = A'^{*}\omega$ and $M' = L(U)$.

2. Elementary Operations

A. Tensor Product: Let U_1 and U_2 be two left Hilbert algebras with completions H_1 and H_2 respectively. Define involution, multiplication and scalar product on $U_1 \otimes U_2$ in the following way :

$$(\xi_1 \otimes \xi_2)(\eta_1 \otimes \eta_2) = (\xi_1\eta_1 \otimes \xi_2\eta_2) \qquad \xi_i, \eta_i \in U_i \ , \quad i = 1,2$$

$$(\xi_1 \otimes \xi_2)^{\#} = \xi_1^{\#} \otimes \xi_2^{\#}$$

$$(\xi_1 \otimes \xi_2 \mid \eta_1 \otimes \eta_2) = (\xi_1 \mid \eta_1)(\xi_2 \mid \eta_2)$$

$U_1 \otimes U_2$ is then a left Hilbert algebra. Denote by $L(U_1)$, $L(U_2)$ & $L(U_1 \otimes U_2)$ the left von Neumann algebra on U_1 , U_2 and $U_1 \otimes U_2$ respectively.

Proposition 2.1 : $L(U_1 \otimes U_2) = L(U_1) \otimes L(U_2)$

Proof. $L(U_1 \otimes U_2)$ and $L(U_1) \otimes L(U_2)$ are both generated by

$$\{\Pi(\xi_1 \otimes \xi_2) = \Pi(\xi_1) \otimes \Pi(\xi_2) \mid \xi_1 \in U_1 \ , \ \xi_2 \in U_2 \}$$

Q. E. D.

Let S_1 , S_2 and S denote the closed extension of the involution on U_1 , U_2 and $U_1 \otimes U_2$ to the Hilbert spaces H_1 , H_2 and $H_1 \otimes H_2$ respectively.

Let $S_1 = J_1\Delta_1^{\frac{1}{2}}$, $S_2 = J_2\Delta_2^{\frac{1}{2}}$ and $S = J\Delta^{\frac{1}{2}}$ be the polar decomposition of S_1 , S_2 and S respectively.

For each densely defined closed operator T on a Hilbert space H with domain $\mathcal{D}(T)$, we consider an inner product in $\mathcal{D}(T)$ defined by :

$$(\xi \mid \eta)_T = (\xi \mid \eta) + (T\xi \mid T\eta) ;$$

$$\|\xi\|_T = (\|\xi\|^2 + \|T\xi\|^2)^{\frac{1}{2}} , \quad \xi, \eta \in \mathcal{D}(T)$$

Because of closedness, $\mathcal{D}(T)$ becomes a Hilbert space with the above inner product. Let $T = UK$ be the polar decomposition of T .

<u>Lemma 2.2</u> A subset M of $\mathcal{D}(T)$ is dense in $\mathcal{D}(T)$ if and only if $(1 + K)M$ is dense in H .

<u>Proof :</u> ([11] lemma 1.1)

Q. E. D.

<u>Lemma 2.3</u> $(1 + \Delta_1^{\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}})(U_1 \otimes U_2)$ is dense in $H_1 \otimes H_2$.

Proof . Define $W \in B(H_1 \times H_2)$ by

$$W = \Delta_1^{\frac{1}{2}}(1+\Delta_1^{\frac{1}{2}})^{-1} \otimes \Delta_2^{\frac{1}{2}}(1+\Delta_2^{\frac{1}{2}})^{-1} + (1+\Delta_1^{\frac{1}{2}})^{-1} \otimes (1+\Delta_2^{\frac{1}{2}})^{-1}$$

Clearly, W is also positive non-singular and hence the range of W is dense in $H_1 \times H_2$.

$$(1+\Delta_1^{\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}})(U_1 \otimes U_2) = W[(1+\Delta_1^{\frac{1}{2}}) \otimes (1+\Delta_2^{\frac{1}{2}})](U_1 \otimes U_2)$$

Since U_1 and U_2 are dense in $\mathcal{D}(S_1)$ and $\mathcal{D}(S_2)$ respectively, $(1+\Delta_1^{\frac{1}{2}})U_1$ and $(1+\Delta_2^{\frac{1}{2}})U_2$ is dense in H_1 and H_2 respectively by lemma 2.2.

$\therefore$ $[(1+\Delta_1^{\frac{1}{2}}) \otimes (1+\Delta_2^{\frac{1}{2}})]\ (U_1 \otimes U_2)$ is dense in $H_1 \otimes H_2$ and hence its image under W is also dense in $H_1 \otimes H_2$.

Q. E. D.

Proposition 2.4. $J = J_1 \otimes J_2$

Proof. From the definitions of S_1 , S_2 and S , it is easily seen that $S = S_1 \otimes S_2$ on $U_1 \otimes U_2$.

(2.1) $J\Delta^{\frac{1}{2}} = (J_1 \otimes J_2)(\Delta_1^{\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}})$ on $U_1 \otimes U_2$. It is clear that $\Delta_1^{\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}}$ is positive symmetric on $U_1 \otimes U_2$ and therefore has a positve self-adjoint extesion Δo in $H_1 \otimes H_2$. ([6], XII.5.1)

By lemma 2.3, $(1+\Delta o)(U_1 \otimes U_2) = (1+\Delta_1^{\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}})(U_1 \otimes U_2)$ is dense in $H_1 \otimes H_2$. By lemma 2.2., $U_1 \otimes U_2$ is dense in $\mathcal{D}(\Delta o)$ and hence Δo is the closure of $\Delta_1^{\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}} \mid U_1 \otimes U_2$ in $H_1 \otimes H_2$. The closure of $J\Delta^{\frac{1}{2}} \mid U_1 \otimes U_2$ is S and hence

$S = (J_1 \otimes J_2)\Delta o$ is an immediate consequence of (2.1)

By the uniqueness of polar decomposition, we have

$$J = J_1 \times J_2$$

$$\Delta^{\frac{1}{2}} = \Delta o = \overline{\Delta_1^{\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}}} \Big|_{U_1 \times U_2}$$

Q. E. D.

Let F_1, F_2 and F be the adjoint operator of S_1, S_2 and S respectively. Then, by lemma 1.2,

$$F_1 = J_1 \Delta_1^{\frac{1}{2}} \quad , \quad F_2 = J_2 \Delta_2^{-\frac{1}{2}} \quad , \quad F = J\Delta^{-\frac{1}{2}} \quad .$$

Let U_1', U_2' and U' denote the set of π'-bounded elements in H_1, H_2 and $H_1 \otimes H_2$ respectively. It is clear that $U_1' \otimes U_2' \subseteq U' \subseteq \mathcal{D}(F)$.

Lemma 2.5. $U_1' \otimes U_2'$ is dense in $\mathcal{D}(F)$.

Proof. By lemma 2.2, it suffices to show that $(1+\Delta^{-\frac{1}{2}})(U_1' \otimes U_2')$ is dense in $H_1 \otimes H_2$.

It is clear that $F = F_1 \otimes F_2$ on $U_1' \otimes U_2'$.

$$J\Delta^{-\frac{1}{2}} = (J_1 \otimes J_2)(\Delta_1^{\frac{1}{2}} \otimes \Delta_2^{-\frac{1}{2}}) \quad \text{on} \quad U_1' \otimes U_2' \; .$$

By proposition 2.4, $J = J_1 \times J_2$.

$$\Delta^{-\frac{1}{2}} = (\Delta_1^{-\frac{1}{2}} \otimes \Delta_2^{\frac{1}{2}}) \quad \text{on} \quad U_1' \otimes U_2' \; .$$

Let $V = \Delta_1^{-\frac{1}{2}}(1+\Delta_1^{-\frac{1}{2}})^{-1} \otimes \Delta_2^{-\frac{1}{2}}(1+\Delta_2^{-\frac{1}{2}}) + (1+\Delta_1^{-\frac{1}{2}})^{-1} \otimes (1+\Delta_2^{-\frac{1}{2}})^{-1}$

Clearly, V is a bounded positive non-singular operator on $H_1 \otimes H_2$ and hence has a dense range in $H_1 \otimes H_2$. Moreover, we have

$$(1+\Delta^{-\frac{1}{2}})(U_1' \otimes U_2') = (1+\Delta_1^{-\frac{1}{2}} \otimes \Delta_2^{-\frac{1}{2}})(U_1' \otimes U_2')$$

$$= V\left[(1+\Delta_1^{-\frac{1}{2}}) \times (1+\Delta_2^{-\frac{1}{2}})\right](U_1' \otimes U_2') .$$

Since U_1' & U_2' are dense in $\mathcal{D}(F_1)$ and $\mathcal{D}(F_2)$ respectively, $(1+\Delta_1^{-\frac{1}{2}})U_1'$ and $(1+\Delta_2^{-\frac{1}{2}})U_2'$ are dense in H_1 and H_2 respectively by lemma 2.2.

Finally, $V\left[(1+\Delta_1^{-\frac{1}{2}}) \times (1+\Delta_2^{-\frac{1}{2}})\right](U_1' \times U_2')$ is dense in $H_1 \otimes H_2$.

Q. E. D.

Lemma 2.6 An element ξ of $H_1 \otimes H_2$ belongs to $\mathcal{D}(S)$ if and only if there exists some $\eta \in H_1 \otimes H_2$ such that

$(\xi \mid \eta_1'^{b} \otimes \eta_2'^{b}) = (\eta_1' \otimes \eta_2' \mid \eta)$ holds for any

$\eta_1' \in U_1'$. $\eta_2' \in U_2'$.

Proof. An immediate consequence of lemma 2.5.

Q. E. D.

Lemma 2.7. An element ξ of $\mathcal{D}(S)$ is Π-bounded if any only if there is a constant $\gamma > 0$ such that

$$\|\Pi'(\eta')\xi\| \leq \gamma \|\eta'\| \quad \text{for any } \eta \in U_1' \otimes U_2' .$$

Proof. Given $\zeta \in U'$, by lemma 2.5 , there exists $\eta_n' \in U_1' \otimes U_2'$ such that $\eta_n \longrightarrow \zeta$

$$\| \Pi(\xi)\eta_n' - \Pi(\xi)\eta_m' \| \leq \gamma \| \eta_n - \eta_m \| \longrightarrow 0$$

By the closedness of $\Pi(\xi)$,

$$\Pi(\xi)\,\zeta = \lim_{n \to \infty} \Pi(\xi)\,\eta_n$$

$$\| \Pi(\xi)\zeta \| = \lim_{n \to \infty} \| \Pi(\xi)\,\eta_n \|$$

$$\leq \gamma \| \eta_n \|$$

$$= \gamma \| \zeta \| \quad .$$

Q. E. D.

Lemma 2.8. $L(U_1 \otimes U_2)'$ is generated by $\Pi'(U_1' \otimes U_2')$.

Proof. Since $U_1' \otimes U_2' \subset U'$, by lemma 1.3 , $L(U_1 \otimes U_2)' =$ $= [\Pi'(U')]''$,

$$L(U_1 \otimes U_2) = [\Pi'(U')]' \subset [\Pi'(U_1' \otimes U_2')]'$$

It therefore suffices to show that $L(U_1 \otimes U_2)$ is dense in $[\Pi'(U_1' \otimes U_2')]'$

Let $X \in \Pi'(U_1' \otimes U_2')$, $\xi_1 , \xi_2\ U_1 \otimes U_2$, $\eta' \in U_1' \otimes U_2'$.

$$(\Pi(\xi_1)\, x\, \xi_2 \mid \eta'^{b}) = (\eta \mid \Pi(\xi_2^{\#})\, x^{*}\, \xi_1^{\#})$$

$\therefore$ $\Pi(\xi_1) \times \xi_2 \in D(S)$ by lemma 2.6 on the other hand,

$$\Pi'(\eta')\,(\Pi(\xi_1) \times \xi_2)$$

$$= (\Pi(\xi_1) \times \Pi(\xi_2))\eta$$

$\Pi(\xi_1) \times \xi_2$ is Π-bounded by lemma 2.7 and

$\Pi(\Pi(\xi_1) \times \xi_2) = \Pi(\xi_1) \times \Pi(\xi_2) \in L(U_1 \otimes U_2)$. ([11] def. 3.2) .

Let $\xi_\alpha \in U_1 \otimes U_2$ $\|\Pi(\xi\ )\| \leq 1$ and $\Pi(\xi_\alpha) \longrightarrow 1$ strongly.
([4] , chap 1, § 3, th. 3) then $\Pi(\xi_\alpha) \times \Pi(\xi_\alpha) \longrightarrow x$ strongly.
$L(U_1 \times U_2)$ is strongly dense in $\Pi'(U_1' \times U_2')'$.

Q. E. D.

<u>Theorem 2.9.</u> $L(U_1 \times U_2)' = L(U_1)' \otimes L(U_2)'$.

<u>Proof.</u> $L(U_1)'$ and $L(U_2)'$ are generated by $\Pi'(U_1')$ and $\Pi'(U_2')$ respectively. (lemma 1.3)

$L(U_1)' \otimes L(U_2)'$ is generated by $\Pi'(U_1') \otimes \Pi'(U_2') =$
$= \Pi'(U_1' \times U_2')$.

The theorem then follows immediately from lemma 2.8

Q. E. D.

As a corollary, the following well-known Theorem of Tomita can be derived from the fact that every von Neumann algebra is isomorphic with a left von Neumann algebra of a left Hilbert algebra.

We shall state the theorem without proof since its proof is a routine application of theorem 3 in section 4 of the first chapter of [4] .

Theorem (Tomita) : Let M_1 and M_2 be two von Neumann algebras, then we have $(M_1 \otimes M_2)' = M_1' \otimes M_2'$

B. Reduction and Induction

Let U be an achieved left Hilbert algebra with completion H and G be a projection in the centre of $L(U)$.

Lemma 2.10. For any $\xi \in U$, $G\xi \in U$, $(G\xi)^{\#} = G\xi^{\#}$ and $\Pi(G\xi) = G\Pi(\xi)$.

Proof. Suppose that $\eta \in U'$,

$$\text{We have } \Pi'(\eta)G\xi = G\,\Pi'(\eta)\xi$$

$$= G\,\Pi(\xi)\eta$$

$$\Pi'(\eta)G\xi^{\#} = G\,\Pi'(\eta)\xi^{\#}$$

$$= G\,\Pi(\xi^{\#})\eta$$

Since $[G\Pi(\xi)]^* = \Pi(\xi)^* G = G\,\Pi(\xi^{\#})$

by ([12] , lemma 2.3) , $G\xi \in U$

$(G\xi)^{\#} = G\,\xi^{\#}$ and $\Pi(G\xi) = G\Pi(\xi)$.

Q. E. D.

<u>Theorem 2.11.</u> GU is a left Hilbert algebra dense in GH and $L(GU) = L(U)G$.

<u>Proof :</u> GU has a scalar product inherited from U and $G\xi \longmapsto G\xi^{\#}$ is clearly an involution on GU (lemma 2.10) .

Condition (I) and (II) in definition (1.1) are valid simply because GU is a subalgebra of U (lemma 2.10) .

Since U^2 is dense in U and

$$\begin{aligned}(GU)^2 &= \{(G\xi)(G\eta) : \xi , \eta \in U\} \\ &= \{G\Pi(\xi)G\eta : \xi , \eta \in U\} \\ &= \{G\xi\eta : \xi , \eta \in U\} \\ &= GU^2 ,\end{aligned}$$

Condition (III) in definition (1.1) is also valid.

For any $\eta' \in U'$, we have

$$\begin{aligned}((G\xi)^{\#} \mid G\eta') &= (G\xi^{\#} \mid \eta') \\ &= (\eta'^{b} \mid G\xi) \\ &= (G\eta'^{b} \mid G\xi) .\end{aligned}$$

Since GU' is dense in H , we see that the involution of GU has a densely defined adjoint operator and hence condition (IV) of definition 1.1 is valid.

$L(GU) = L(U)G$ is obvious by lemma 2.10.

Q. E. D.

Lemma 2.12. $(GU)' = GU'$

Proof : It is clear that $GU' \subseteq (GU)'$.

Suppose $x \in (GU)' \subseteq GH$, then there exists some $y \in GH$ such that given any $\xi \in U$, we have

$$(\xi \mid x) = (G\xi \mid x) = (y \mid G\xi^{\#}) = (y \mid \xi^{\#}) .$$

Also

$$\Pi(\xi)x = \Pi(\xi)Gx = \Pi(G\xi)Gx$$

and hence $\xi \to \Pi(\xi)x$ is bounded.

$\therefore$ $x \in GU'$ and $(GU)' \subseteq GU'$.

Q. E. D.

Lemma 2.13. GU is achieved .

Proof : by lemma 2.12 , $(GU)'' = [(GU)']' = [GU']' = GU'' = GU$.

Q. E. D.

Theorem 2.11 states that the reduction (or induction) of the left von Neumann algebra of a left Hilbert algebra by a central projection is again the left von Neumann algebra of a certain left Hilbert algebra. This assertion is false if the projection is not central as is shown by the following example.

Example 2.14 . Let U be the algebra of all two by two complex matrices. Let $A^{\#} = A^{*}$ for any $A \in U$. A scalar product is defined on U in the following way :

$$(A \mid B) = \mathrm{tr}(B^* A) \quad \text{for} \quad A , B \in U ,$$

where

$$\mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

U is easily seen to be a Hilbert algebra which is complete with $\Pi(U) = L(U) \simeq B(H_2)$

Let

$$E = \Pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right), \quad \text{then } E \text{ is a projection of } L(U) ,$$

$$L(U)_E = \left\{\Pi\left(\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right) : x \in C \right\} \quad \text{and} \quad E(U) = \left\{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in C \right\}$$

It is clear that the reduction of L(U) by E does not have a cyclic vector and hence is not the left von Neumann algebra of any left Hilbert algebra (see lemma 3.4 and the remarks following it) .

It follows that the induction of $L(U)'$ by E is not the left von Neumann algebra of any left Hilbert algebra since $L(U)_E = (L(U)'_E)'$.

3. A Characterization

Let U be an achieved Hilbert algebra with completion H and $L(U)$ the left von Neumann algebra of U .

From theorem 2.11 , GU is an achieved left Hilbert algebra dense in $G(H)$. Let S_1 be the closed extension of the involution of GU to GH and $S_1 = J_1\Delta_1^{\frac{1}{2}}$ be the polar decomposition of S_1 . Similarly, $(1 - G)U$ is an achieved left Hilbert algebra dense in $(1 - G)H$. Let S_2 be the closed extension of the involution of $(1 - G)U$ to $(1 - G)H$ and $S_2 = J_2\Delta_2^{\frac{1}{2}}$ the polar decomposition of S_2 .

Lemma 3.1. Let S be the closed extension of the involution of U to H and $S = J\Delta^{\frac{1}{2}}$ the polar decomposition of S . Let G be a projection in the centre of $L(U)$. We have $JGJ = G$.

Proof. Suppose that $\xi \in U$. By lemma 3.1

$$
\begin{aligned}
S\,\xi &= \xi^{\#} \\
&= G\xi^{\#} + (1 - G)\xi^{\#} \\
&= (G\xi)^{\#} + ((1 - G)\xi)^{\#} \\
&= S_1 G\xi + S_2(1 - G)\xi
\end{aligned}
$$

Suppose that $\xi \in D(S)$, then there exists $\xi_n \in U$ such that $\xi_n \longrightarrow \xi$ and $\xi_n^{\#} \longrightarrow S\,\xi$

$$S\xi = \lim_{n \to \infty} \xi_n^{\#}$$

$$= \lim_{n \to \infty} (G\xi_n)^{\#} + \lim_{n \to \infty} ((1 - G)\xi_n)^{\#}$$

$$= S_1(G\xi) + S_2((1 - G)\xi)$$

$$S \subseteq S_1 G + S_2(1 - G)$$

The inverse inclusion $S \supseteq S_1 G + S_2(1 - G)$ can be proved in a similar way.

$$\therefore \quad S = S_1 G + S_2(1 - G)$$

$$J\Delta^{\frac{1}{2}} = J_1 \Delta_1^{\frac{1}{2}} G + J_2 \Delta_2^{\frac{1}{2}}(1 - G)$$

$$= (J_1 G + J_2(1 - G))(\Delta_1^{\frac{1}{2}} G + \Delta_2^{\frac{1}{2}}(1 - G))$$

It is clear that $J_1 G + J_2(1 - G)$ is an unitary involution and $\Delta_1^{\frac{1}{2}} G + \Delta_2^{\frac{1}{2}}(1 - G)$ is a positive operator.

By the uniqueness of polar decomposition, we have

$$J = J_1 G + J_2(1 - G)$$

$$JGJ = JG(J_1 G + J_2(1 - G))$$

$$= (J_1 G + J_2(1 - G))J_1 G$$

$$= (J_1 G)(J_1 G)$$

$$= J_1^2 G$$

$$= G$$

Q. E. D.

<u>Corollary 3.2.</u> $JAJ = A^*$ for any A $L(U) \cap L(U)'$

<u>Proof.</u> Suppose first that $A = A^*$,

then A is the uniform limit of a sequence of operators $\{S_i\}_{i=1}^{\infty}$ in $L(U) \cap L(U)'$, where

$$S_i = \sum_{k=1}^{n_i} \alpha_k P_k \;, \quad \alpha_k \in \mathbb{R} \;, \quad P_k \text{ is a central}$$

projection for $k = 1, \ldots, n_i$.

Since

$$JS_iJ = \sum_{k=1}^{n_i} \alpha_k JP_kJ$$

$$= \sum_{k=1}^{n_i} \alpha_k P_k \quad \text{(lemma 3.2)}$$

$$= S_i$$

We have

$$JAJ = A .$$

Now any $A \in L(U) \cap L(U)'$ can be written as $A_1 + iA_2$ with $A_1 = A_1^*$, $A_2 = A_2^*$, A_1, $A_2 \in L(U) \cap L(U)'$

$$JAJ = JA_1J - i\,JA_2J$$

$$= A_1 - i\,A_2$$

$$= A^*$$

Q. E. D.

Let M be a von Neumann algebra on H with a cyclic and separating vector ω . Consider the left Hilbert algebra U defined in example 1.6. Lemma 3.1 can be directly verified in this case.

Let G be a projection in the centre of M

$$J\Delta^{\frac{1}{2}}G\omega = SG\omega = G\omega$$

$$\Delta^{\frac{1}{2}}G\omega = JG\omega$$

$$J\Delta^{-\frac{1}{2}}G\omega = FG\omega = G\omega$$

$$\Delta^{-\frac{1}{2}}G\omega = JG\omega$$

$$\Delta^{\frac{1}{2}}G\omega = \Delta^{-\frac{1}{2}}G\omega$$

$$\Delta G\omega = G\omega$$

$$(1+\Delta^{\frac{1}{2}})(1-\Delta^{\frac{1}{2}})G\omega = 0$$

$$(1-\Delta^{\frac{1}{2}})G\omega = 0$$

$$\text{or} \quad \Delta^{\frac{1}{2}}G\omega = G\omega$$

$$J\,G\omega = G\omega$$

in particular, when $G=1$, $J\omega = \omega$.

$$\therefore \quad (JGJ)\omega = G\omega$$

Since $G\,,JGJ \in M'$ and ω is separating for M' , $JGJ = G$.

Definition 3.3. A von Neumann algebra $[M, H]$ is said to be standard if there exists a unitary involution J on H such that

(i) $J M J = M'$

(ii) J commutes with every central projection in M.

(c.f. [13])

Lemma 3.4. Let M be a standard von Neumann algebra which is σ-finite, then M possesses a cyclic and separating vector.

Proof. ([4] , Chap. III, § 1 , lemma 5)

Q. E. D.

From lemmas 1.4 and 3.1, $L(U)$ is standard for every left Hilbert algebra. Example 2.14 shows that the reduction (or induction) of a standard von Neumann algebra is not necessarily standard.

Lemma 3.5. Let M (resp. M_1) be standard von Neumann algebras, then any isomorphism Φ from M onto M , is spatial.

Proof. ([4] , Chap III , § 1 , th. 6)

Q. E. D.

Theorem 3.6. Let M be a von Neumann algebra. In order that $M = L(U)$ for some left Hilbert algebra U , it is necessary and sufficient that M is standard.

Proof. Necessity is clear.

Conversely, if M is standard, since $M \simeq L(U)$ for some left Hilbert algebra U, M is spatially isomorphic with $L(U)$ by lemma 3.5.

Q. E. D.

4. Weight, Modular Automorphism group and K.M.S. condition

Definition 4.1. Let M be a von Neumann algebra. A function ϕ defined on M^+ with range in $[0, +\infty]$ is called a weight if the following conditions are satisfied.

(i) $\phi(x+y) = \phi(x) + \phi(y)$ $\quad x,y \in M^+$

(ii) $\phi(\lambda x) = \lambda\phi(x)$ $\quad \lambda \in \mathbb{R}^+, \ x \in M^+$

(we take the convention that $0(+\infty) = 0$).

ϕ is called normal if there is a set $\{\omega_i\}$ of normal positive functionals on M such that

$$\phi(x) = \sup \omega_i(x) \quad \text{for each } x \text{ in } M^+$$

Let $\mathfrak{n}_\phi$ denote the set $\{x \in M : \phi(x^*x) < \infty\}$

Let $\mathfrak{m}_\phi^+$ denote the set $\{x \in M^+ : \phi(x) < \infty\}$

Let $\mathfrak{m}_\phi$ denote the linear span of $\mathfrak{m}_\phi^+$ and there is a natural extension of ϕ to a positive linear functional on $\mathfrak{m}_\phi$

(again denoted by ϕ) .

ϕ is called semi-finite if $\mathfrak{M}_\phi$ is σ-weakly dense in M

ϕ is called faithful if $\phi(x) = 0$ implies $x = 0$ for each x in M^+ .

It is clear that $\mathfrak{N}_\phi$ is a left ideal of M .

<u>Lemma 4.2.</u> $\mathfrak{M}_\phi = \mathfrak{N}_\phi^* \mathfrak{N}_\phi$

<u>Proof.</u> ([1] lemma 1.3)

Q. E. D.

<u>Lemma 4.3.</u> $\mathfrak{M}_\phi = (\mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*)^2$

<u>Proof.</u> By lemma 4.2, $\mathfrak{M}_\phi = \mathfrak{N}_\phi^* \mathfrak{N}_\phi$.

$\therefore$ $(\mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*)^2 \subseteq \mathfrak{M}_\phi$. On the other hand, for each $\chi \in \mathfrak{M}_\phi^+$, we have $\chi^{\frac{1}{2}} \in \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*$ and hence $\chi \in (\mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*)^2$. We have

$$\mathfrak{M}_\phi = \text{lin}\, \mathfrak{M}_\phi^+ \subset (\mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*)^2 .$$

Q. E. D.

Suppose now that ϕ is a semi-finite faithful normal weight on the positive part M^+ of a von Neumann algebra M .

For $x, y \in \mathfrak{N}_\phi$, define an inner product $(x|y) = \phi(y^* x)$.

It is clear that with this inner product $\mathfrak{N}_\phi$ is T_2 and the completion H_ϕ of $\mathfrak{N}_\phi$ is a Hilbert space.

Let Λ_ϕ be the canonical injection from $\mathfrak{N}_\phi$ into H_ϕ .

For each $x \in M$, the map $\Lambda_\phi(y) \longmapsto \Lambda_\phi(xy)$ is continuous for $y \in \mathcal{N}_\phi$ and hence can be extended in a unique way to a bounded operator on H_ϕ, which we shall denote by $\Pi_\phi(x)$. Clearly Π_ϕ is a *-representation of M on H_ϕ.

<u>Lemma 4.4</u>. Π_ϕ is a faithful, non-dengerate *-representation of M on H_ϕ.

<u>Proof.</u> Suppose that $x \in \mathcal{N}_\phi$. Let $\{U_i\}$ be a net of approximate unit in $\mathcal{N}_\phi$ ([5], 1.7.2)

$$\|\Lambda_\phi(x) - \Lambda_\phi(U_i x)\|^2 = \phi[(x-U_i x)^*(x-U_i x)]$$

$$\leqq 2[\phi(x^*x) - \phi(x^*U_i x)] \to 0$$

Since $\|x^*U_i - x^*\| \to 0$ and ϕ is uniformly lower-semicontinuous on M^+

$$\Pi_\phi(M)[\Lambda_\phi(\mathcal{N}_\phi)] \supseteq \Pi_\phi(\{U_i\})[\Lambda_\phi(\mathcal{N}_\phi)]$$

$\therefore$ Π_ϕ is non-degenerate.

Suppose that $\Pi_\phi(x) = 0$ for some $x \in \mathcal{N}_\phi$. Then, we have

$$\Pi_\phi(x^*x) = 0$$

i.e. $(\Pi_\phi(x^*x)\Lambda_\phi(y) \mid \Lambda_\phi(z)) = 0 \qquad y, z \in \mathcal{N}_\phi$

$$\phi(z^* x^* x y) = 0 \qquad y, z \in \mathcal{N}_\phi$$

Since $\mathcal{N}_\phi$ is σ-weakly (and hence σ-strongly) dense in M, there exists $y_\lambda \in \mathcal{N}_\phi \cap M^+$ such that $y_\lambda \longrightarrow 1$ σ- strongly.

We have $\phi(y_\lambda x^* x y_\lambda) = 0$ and $y_\lambda x^* x y_\lambda \longrightarrow x^* x$

σ-weakly, by the σ-weakly lower-semicontinuity of ϕ on M^+,

$0 \leq \phi(x^*x) \leq \lim \phi(y_\lambda x^*x y_\lambda) = 0$

$$\phi(x^*x) = 0$$

which implies $\qquad x^*x = 0 \qquad (\phi$ is faithful)

and therefore $\qquad x = 0$.

Q. E. D.

<u>Lemma 4.5.</u> Let $U = \Lambda_\phi(\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*)$. Then U^2 is dense in $\Lambda_\phi(\mathfrak{n}_\phi)$. In particular, U is dense in H_ϕ .

<u>Proof.</u> Let $x \in \mathfrak{n}_\phi$. Let $\{U_i\}$ be a net of approximate unit in $\mathfrak{n}_\phi$. As in the proof of lemma 4.4, we have $\Lambda_\phi(U_i x) \longrightarrow \Lambda_\phi(x)$ and note that $U_i x \in \mathfrak{n}_\phi^* \mathfrak{n}_\phi = \mathfrak{m}_\phi = (\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*)^2$.

Q. E. D.

From lemma 4.4, we see that M is isomorphic with $\Pi_\phi(M)$ on H_ϕ . F. Combes proved that $\Pi_\phi(M)$ is the left von Neumann algebra of a left Hilbert algebra dense in H_ϕ . (see [2]) .

<u>Theorem 4.6.</u> Let M be a von Neumann algebra, ϕ a semi-finite faithful normal weight on M^+ . Let $U = \Lambda_\phi(\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*)$, and define an inner product on U by $(\Lambda_\phi(x) \mid \Lambda_\phi(y)) = \phi(y^* x)$. Define multiplication and involution on U by $(\Lambda_\phi(x))(\Lambda_\phi(y)) = \Lambda_\phi(x y)$ and $\Lambda_\phi(x)^\# = \Lambda_\phi(x^*)$ respectively. Then U is an achieved left Hilbert algebra dense in H_ϕ . For $x \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$,

we have $\Pi(\Lambda_\phi(x)) = \Pi_\phi(x)$ such that $L(U) = \Pi_\phi(M)$.

Proof. ([2] , th. 2.13.)

Q. E. D.

On the other hand, given a left Hilbert algebra, a certain canonical weight can be defined on it.

Definition 4.7. Let U be a left Hilbert algebra with completion H . An element $\xi \in H$ is called left-bounded if the map $\eta \mapsto \Pi'(\eta)\xi$ defines a bounded linear operator on U' . This operator can be extended to a bounded operator on H which we shall denote by $\Pi(\xi)$. Clearly $\Pi(\xi) \in L(U)$. Let H_g denote the set of all left-bounded elements in H .

Theorem 4.8. Let U be a left Hilbert algebra with completion H . For $x \in L(U)^+$, let $\phi(x) = (\alpha \mid \alpha)$ if there exists $\alpha \in H_g$ such that $x^{\frac{1}{2}} = \Pi(\alpha)$, otherwise let $\phi(x) = +\infty$.

Then ϕ is a semi-finite faithful normal weight on $L(U)^+$. The left ideal $\mathfrak{n}_\phi$ is identical with the left ideal $\{\Pi(\alpha) : \alpha \in H_g\}$ and for $\alpha, \beta \in H_g$, we have $\phi(\Pi(\beta)^* \Pi(\alpha)) = (\alpha \mid \beta)$.

The * subalgebra $\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$ of $L(U)$ is identical with $\Pi(U'')$. The * subalgebra $\mathfrak{m}_\phi = \mathfrak{n}_\phi^* \mathfrak{n}_\phi$ of $L(U)$ is identical with $\Pi((U'')^2)$. ϕ is invariant under the modular automorphism group of $L(U)$.

Furthermore, if ψ is a faithful normal semi-finite weight on the positive part M^+ of a von Neumann algebra M . Let ϕ be the canonical weight on $L(U)^{+} = \Pi_{\phi}(M)^{+}$. We have $\psi(x) = \phi(\Pi_{\phi}(x))$ for all $x \in M^{+}$.

Proof. ([2] , th. 2.11)

Q. E. D.

Definition 4.9. Let M be a von Neumann algebra, ϕ a weight on M^{+}. σ_t a one parameter group of autormorphisms of M. We say that ϕ is a K.M.S. weight with respect to σ_t if the following two conditions are satisfied.

(i) ϕ is invariant under σ_t, for all $t \in \mathbb{R}$

i.e. $\phi(\sigma_t(x)) = \phi(x)$, $t \in \mathbb{R}$, $x \in M^{+}$.

(ii) For any pair (a, b) of elements of $\mathfrak{n}_{\phi} \cap \mathfrak{n}_{\phi}^{*}$, there exists a function F which is continuous and uniformly bounded on the band $0 \leq \operatorname{Im} Z \leq 1$ in $\mathbb{C}$. F is also assumed to be holomorphic inside this band and such that

$$F(t) = \phi[\sigma_t(a)b], \quad F(t+i) = \phi[b\,\sigma_t(a)]$$

for all $t \in \mathbb{R}$.

Lemma 4.10. Let H be a Hilbert space, h a densely defined self-adjoint operator on H. Let $s > 0$ be a real number. Then for each $\xi \in (\exp(sh))$, the function $Z \longmapsto \exp(Zh)\xi$ is defined, continuous, uniformly bounded on the band $0 \leq \operatorname{Re} Z \leq s$ and is holomorphic inside the band.

Proof. ([2] , lemma 2.6)

Q. E. D.

Lemma 4.11. Let U be a left Hilbert algebra. For any $\xi , \eta \in D^{\#}$, there exists a function F defined, continuous and bounded on the band $0 \leq \operatorname{Im} Z \leq 1$. F is holomorphic inside this band and that

$$F(t) = (\xi \mid \Delta^{it} \eta^{\#})$$

$$F(t+1) = (\Delta^{it} \eta \mid \xi^{\#}) \qquad \text{for all} \quad t \in \mathbb{R} .$$

Proof. ([2] , lemma 4.3)

Q. E. D.

The proof of the following theorem was sketched by F. Combes in [2] and we give the details here for the sake of completeness.

Theorem 4.12. Let M be a von Neumann algebra, ϕ a semi-finite faithful normal weight on M^{+} . Then there exists a one-parameter group σ_t of automorphisms of M such that ϕ is a K.M.S. weight w.r.t. σ_t .

Proof. By theorem 4.6, ϕ defined a left Hilbert algebra U on H_ϕ and M is isomorphic with $\Pi_\phi(M) = L(U)$. Let Δ be the modular operator on H_ϕ w.r.t. U . Define $\sigma_t(x)$ by

$$\Pi_\phi(\sigma_t(x)) = \Delta^{it} \Pi_\phi(x) \Delta^{-it}$$

By theorem 4.8, $\phi(\sigma_t(x)) = \tau(\Delta^{it}\Pi_\phi(x)\Delta^{-it})$

$$= \tau(\Pi_\phi(x))$$

$$= \phi(x) .$$

where τ is the canonical weight defined on $L(U)$.

Let $x, y \in \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*$ and $\xi = \Lambda_\phi(x)$, $\eta = \Lambda_\phi(y)$.

By lemma 4.11, there exists a function F defined on $0 \leq \operatorname{Im} Z \leq 1$. F is continuous and uniformly bounded on the band $0 \leq \operatorname{Im} Z \leq 1$ and holomorphic inside it ,

$$F(t) = (\xi \mid \Delta^{it}\eta^{\#})$$

$$= \tau(\Pi(\Delta^{it}\eta)\Pi(\xi)) \qquad \text{(th. 4.8)}$$

$$= \tau(\Delta^{it}\Pi(\eta)\Delta^{-it}\Pi(\xi)) \qquad \text{(lemma 1.5)}$$

$$= \tau(\Pi_\phi(\sigma_t(y))\Pi_\phi(x))$$

$$= \phi(\sigma_t(y)\chi)$$

$$F(t+i) = (\Delta^{it}\eta \mid \xi^{\#})$$

$$= \tau(\Pi(\xi)\Pi(\Delta^{it}\eta))$$

$$= \tau(\Pi_\phi(x)\Pi(\sigma_t(y)))$$

$$= \phi(\chi\,\sigma_t(y)) .$$

Q. E. D.

As an immediate consequence of theorem 4.12, we have the following theorem of Takesaki. ([11] , th. 13.1)

Corollary 4.13. Let M be a von Neumann algebra and ϕ a normal fiathful linear functional of M . Then there exists a one-parameter automorphism group σ_t of M such that ϕ_o satisfies the K.M.S. condition.

Consider a von Neumann algebra M and ϕ a semi-finite faithful normal weight on M^+ . By theorem 4.12, there exists a one-parameter automorphism group σ_t of M such that ϕ is a K.M.S. weight w.r.t. σ_t . Suppose now that α_t is another one-parameter automorphism group of M such that ϕ is a K.M.S. weight w.r.t. α_t . We are now going to prove that $\alpha_t = \sigma_t$. This uniqueness theorem was proved by F. Combes in [2] , but the proof there is intrinsically involved with modular Hilbert algebra. The method we employed here is a modification of the ideas in [10] which is more direct, i.e. making no use of modular Hilbert algebra.

Lemma 4.14. There exists a unique one-parameter strongly continuous unitary group V_t on H_ϕ such that

$$V_t(\Lambda_\phi(x)) = \Lambda_\phi(\alpha_t(x)) \qquad \text{for all} \quad x \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$$

Proof. Since $\Lambda_\phi(\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*)$ is dense in H_ϕ (lemma 4.5) , uniqueness is clear.

For each $t \in \mathbb{R}$, define $V_t(\Lambda_\phi(x)) = \Lambda_\phi(\alpha_t(x))$ for $x \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$. (note that $\alpha_t(\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*) = (\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*)$.

$$\|V_t(\Lambda_\phi(x))\|^2 = \big(\Lambda_\phi(\alpha_t(x)) \mid \Lambda_\phi(\alpha_t(x))\big)$$

$$= \phi(\alpha_t(x^*)\alpha_t(x))$$

$$= \phi(x^*x) \qquad (\phi \text{ is invariant under } \alpha_t)$$

$$= \|\Lambda_\phi(x)\|^2$$

V_t can be extended to a unitary operator on H_ϕ which we shall denote again by V_t .

$$V_t \cdot V_s(\Lambda_\phi(x)) = V_t \Lambda_\phi(\alpha_s(x))$$

$$= \Lambda_\phi\big(\alpha_t(\alpha_s(x))\big)$$

$$= \Lambda_\phi(\alpha_{t+s}(x))$$

$$= V_{t+s}\Lambda_\phi((x))$$

$\therefore$ V_t is a one-parameter group.

The function $t \longmapsto (V_t(\Lambda_\phi(x)) \mid \Lambda_\phi(y))$

$$= (\Lambda_\phi(\alpha_t(x)) \mid \Lambda_\phi(y))$$

$$= \phi\left[y^*\alpha_t(x)\right]$$

is continuous by the K.M.S. condition. $\therefore$ V_t is weakly (and hence strongly) continuous.

Q. E. D.

<u>Lemma 4.15.</u> Given $x, y \in \mathcal{M}_\phi$, $x = x^*$, $y = y^*$, there exists a function f defined on the band $0 \leq \mathrm{Im}\, Z \leq 1$, continuous and uniformly bounded on the band, holomorphic inside the band and such that

$$f(t) = (V_t \Lambda_\phi(x) \mid \Lambda_\phi(y))$$

$$f(t+i) = (\Lambda_\phi(y) \mid V_t \Lambda_\phi(x)) = \overline{f(t)}$$

<u>Proof.</u> By the K.M.S. condition, there exists a uniformly bounded function f defined on $0 \leq \mathrm{Im}\, Z \leq 1$, continuous on and holomorphic inside this band such that

$$\begin{aligned} f(t) &= \phi[y\, \sigma_t(x)] \\ &= (\Lambda_\phi(\sigma_t(x)) \mid \Lambda_\phi(y)) \\ &= (V_t(\Lambda_\phi(x)) \mid \Lambda_\phi(y)) \\ f(t+i) &= \phi[\sigma_t(x)\, y] \\ &= (\Lambda_\phi(y) \mid \Lambda_\phi(\sigma_t(x))) \\ &= (\Lambda_\phi(y) \mid V_t(\Lambda_\phi(x))) \end{aligned}$$

Q. E. D.

<u>Lemma 4.16.</u> Let $\xi , \eta \in \overline{\{ \Lambda_\phi(x): \ x = x^* , \ x \in \mathcal{M}_\phi \}}$ then there exists a function f with the same properties as those stated in lemma 4.15 with x , y replaced by ξ , η .

Proof. Let $x_n , y_n \in \mathfrak{M}_\phi$, $x_n^* = x_n$, $y_n^* = y_n$,
and $\Lambda_\phi(x_n) \longrightarrow \xi$, $\Lambda_\phi(y_n) \longrightarrow \eta$.

By lemma 4.15 , there exists f_n with the properties stated and we have

$$\|f_n - f_m\| \leq \|\Lambda_\phi(x_n) - \Lambda_\phi(x_m)\| \, \|\Lambda_\phi(y_n)\|$$
$$+ \|\Lambda_\phi(x_n)\| \, \|\Lambda_\phi(y_n) - \Lambda_\phi(y_m)\|$$
$$\longrightarrow 0 \quad \text{as} \quad n , m \longrightarrow \infty$$

Let f be the limit of f_n , then f has the required properties.

Q. E. D.

Definition 4.17. The function f in lemma 4.16 shall be called the function associated with V_t and (ξ , η) .

Lemma 4.18. The function f associated with V_t , (ξ , η) has the following property.

$$f(t) = (V_t \xi | \eta)$$

$$f(t + \tfrac{i}{2}) \quad \text{is real for all real } t$$

Proof. Define g on $0 \leq \operatorname{Im} Z \leq 1$ by

$$g(Z) = \overline{f(\overline{z} + i)}$$

g satisfies same conditions as f and hence $f = g$ on $0 \leq \operatorname{Im} Z \leq 1$ which implies

$$\overline{f(t+\frac{i}{2})} = g(t+\frac{i}{2}) = f(t+\frac{i}{2})$$

Q. E. D.

<u>Definition 4.19.</u> Let $K = \{\overline{\Lambda_\phi(x) : x \in \mathfrak{M}_\phi , x = x^*}\}$

It is clear that K is total in H_ϕ for $\mathrm{lin}\, K \supseteq U^2$, and U^2 is dense in H by lemma 4.5.

<u>Lemma 4.20.</u> Let $x , y \in \mathfrak{M}_\phi \cap M^s$,

then $(\Lambda_\phi(x) \mid J\Lambda_\phi(y)) = (J\Lambda_\phi(y) \mid \Lambda_\phi(x)) \in \mathbb{R}$.

By continuity, the above identity also holds for $\xi , \eta \in K$ in place of X , y .

<u>Proof.</u> By lemma 4.5 , x , y can be represented by real linear combinations of elements of the form z^*z with $z \in \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*$.

It suffices to show that

$$(\Lambda_\phi(z^*z) \mid J\Lambda_\phi(t^*t)) = (J\Lambda_\phi(t^*t) \mid \Lambda_\phi(z^*z))$$

for all $t , z \in \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*$

$$\begin{aligned}(\Lambda_\phi(z^*z) \mid J\Lambda_\phi(t^*t)) &= (\Pi_\phi(z)^*\Lambda_\phi(z) \mid J\Lambda_\phi(t)^*\Lambda_\phi(t)) \\ &= (\Lambda_\phi(z) \mid J\Pi'(J\Lambda_\phi(z))\Pi_\phi(t)^*\Lambda_\phi(t)) \\ &= (\Lambda_\phi(z) \mid J\Pi_\phi(t^*t)J\Lambda_\phi(z)) \\ &= (\Pi'(J\Lambda_\phi(t^*t)\Lambda_\phi(z) \mid \Lambda_\phi(z)) \\ &= (\Pi_\phi(z)J\Lambda_\phi(t^*t) \mid \Lambda_\phi(z))\end{aligned}$$

$$= (J\Lambda_\phi(t^*t) \mid \Lambda_\phi(z^*z))$$

Q. E. D.

Definition 4.21. $\eta \in H_\phi$ is said to be an entire vector for V_t if there is an entire H_ϕ- valued function, h, such that $h(t) = V_t(\eta)$ for all real t.

Lemma 4.22. The set of entire vectors of K is dense in K.

Proof. Since $\alpha_t(\mathcal{M}_\phi) = \mathcal{M}_\phi$, it is easily seen that $V_t K \subseteq K$.

for $\eta \in K$ let

$$\eta_n = \left(\frac{n}{\Pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-nt^2} V_t \eta \, dt$$

Then $\eta_n \in K$ and $\eta_n \longrightarrow \eta$ as $\eta \to \infty$

Also

$$h_n(z) = \left(\frac{n}{\Pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} V_t \eta \, dt$$

is an entire function with $h_n(t) = V_t(\eta_n)$.

Q. E. D.

Theorem 4.23. $\alpha_t = \sigma_t$ for all $t \in \mathbb{R}$

Proof. Let η be an entire vector for $\{ V_t \}$ in K . and let h be the entire function such that $h(t) = V_t(\eta)$ for real t . Then for fixed real t the functions $h(t + i z)$ and $V_t(h(i z))$ are both entire, and agree for purely imaginary z . They are thus equal, and in particular

$$h(t + i s) = V_t(h(i s)) \quad \text{for all real } s \text{ and } t .$$

Since V_t is unitary, $h(z)$ is bounded on $0 \leq \operatorname{Im} Z \leq 1$.

Let $x \in \mathcal{M}_\phi \cap M^s$ and $\xi = \Lambda_\phi(x)$, then $\xi \in D(\Delta^{\frac{1}{2}})$ and $\xi = \xi^\#$. We would like to show that

$$(V_t \eta \mid \Delta^{it} J\xi) = (\eta \mid J \xi)$$

for this will then imply that $V_t = \Delta^{it}$ since K and $J(\mathcal{M}_\phi \cap M^s)$ are total in H_ϕ .

Define $g(z) = (h(z) \mid J\Delta^{-iz} \xi)$

By lemma 4.10, g is holomorphic in $0 \leq \operatorname{Im} Z \leq \frac{1}{2}$. Also g is continuous on $0 \leq \operatorname{Im} Z \leq 1/2$.

$$g(t) = (V_t \eta \mid J\Delta^{it} \xi)$$

$$g(t + \tfrac{i}{2}) = (h(t - \tfrac{i}{2}) \mid \Delta^{it} \xi)$$

On $0 \leq \operatorname{Im} Z \leq 1/2$, $\Delta^{-iz} = \Delta^{-it} \Delta^s \xi$ with $0 \leq S \leq \frac{1}{2}$ and hence $\|\Delta^{-iz} \xi\| = \|\Delta^s \xi\|$ is bounded . Since $h(z)$ is bounded on $0 \leq \operatorname{Im} Z \leq \frac{1}{2}$, $g(z)$ is therefore bounded on $0 \leq \operatorname{Im} Z \leq \frac{1}{2}$.

By lemma 4.20, (note that $\Delta^{it}\xi = \Lambda_\phi(\sigma_t(x))$ and $\sigma_t(x) \in \mathcal{M}_\phi \cap M^s$), $g(t)$ is real for all real t.

On the other hand, by lemma 4.16, there exists a function f associated with V_t, $(\eta, \Delta^{is}\xi)$. By lemma 4.18, $f(t + \frac{i}{2})$ is real and

$$f(t) = (V_t\eta \mid \Delta^{is}\xi) = (h(t) \mid \Delta^{is}\xi) \quad \text{for all real } t .$$

Thus f agrees on X-axis with the entire function $(H(z) \mid \Delta^{is}\xi)$, and so must agree with it every where on the strip $0 \leq \operatorname{Im} Z \leq \frac{1}{2}$. It follows that $(h(z) \mid \Delta^{is} z)$ is real for $z = t + \frac{i}{2}$. In particular this must be true for $s = t$, so that $g(t + \frac{i}{2})$ is real for all real t.

Thus g is defined, continuous and bounded on the strip $o \leq \operatorname{Im} Z \leq \frac{1}{2}$, holomorphic in the interior, and real-valued on the boundary. By repeated applications of the Schwartz reflection principle and Lionville's Theorem, g must be reduced to a constant.

$$(V_t\eta \mid J\Delta^{it}\xi) = (\eta \mid J\xi) \quad \text{for all real } t .$$

$$\therefore \quad V_t = \Delta^{it}$$

Finally $\Pi_\phi(\alpha_t(x))(\Lambda_\phi(y)) \qquad x, y \in \eta_\phi \cap \eta_\phi^*$

$$= \Lambda_\phi(\alpha_t(x)\, y)$$

$$= \Lambda_\phi(\alpha_t(x \cdot \alpha_t(y)))$$

$$= V_t\, \Lambda_\phi(x \cdot \alpha_t(y)$$

$$= \left(V_t \, \Pi_\phi(x)\right) \left(\Lambda_\phi(\alpha_{-t}(y))\right)$$

$$= \left(V_t \, \Pi_\phi(x) \, V_{-t}\right) \left(\Lambda_\phi(y)\right)$$

$$\because \quad \Pi_\phi(\sigma_t(x)) = \Delta^{it} \, \Pi_\phi(x) \, \Delta^{-it}$$

$$\therefore \quad \Pi_\phi(\sigma_t(x)) = \Pi_\phi(\alpha_t(x))$$

$$\therefore \quad \sigma_t(x) = \alpha_t(x) \qquad x \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$$

Suppose $x \in M$, let $x_\lambda \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$ and $X_\lambda \longrightarrow x$ σ-weakly, since σ_t and α_t are σ-weakly continuous,

$$\alpha_t(x) = \lim \alpha_t(x_\lambda) = \lim \sigma_t(x_\lambda) = \sigma_t(x)$$

$$\alpha_t(x) = \sigma_t(x) \qquad \text{for all} \quad t \in \mathbb{R}, \quad x \in M.$$

Q. E. D.

As an immediate corollary, we have the following theorem of Takesaki ([11] , Th. 13.2)

<u>Corollary 4.24.</u> Let M be a von Neumann algebra. Let σ_t be a one-parameter automorphism group of M. If a faithful normal positive linear functional ϕ_o of M satisfies the K.M.S. condition w.r.t. σ_t, then σ_t is the modular automorphism group of M associated with ϕ_o.

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